ON THE PROJECTIVE DIFFERENTIAL GEOMETRY
OF CUBIC RULED SURFACES.

by

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I. The General Methods of Projective Differential Geometry.

The surface generated by a straight line moving through space is called a ruled surface, and the line in its various positions is called a generator. If the generators are all tangent to a fixed curve the surface is said to be developable; otherwise it is said to be skew or non-developable. Upon a ruled surface can be drawn any number of curves which meet every generator in one point. Any two such curves can be put into one-to-one point correspondence by calling points on the same generator corresponding points. This correspondence can be shown analytically by expressing each curve parametrically in such a way that points on the same generator are given by the same value of the parameter.

E. J. Wilczyhski has developed a method for the study of non-developable ruled surfaces by means of systems of differential equations. The general form of the systems is

$$
\text{(A)} \quad \begin{aligned}
y'' + p_{11} y' + p_{12} z' + q_{11} y + q_{12} z &= 0 \\
z'' + p_{21} y' + p_{22} z' + q_{21} y + q_{22} z &= 0,
\end{aligned}
$$

where the primes indicate derivatives of the dependent variables y and z with respect to x, and where the coefficients are functions of x. Such a system of differential equations ~~) defines two functions y and z of x, which are analytic in the vicinity of $x = x_0$, if the coefficients are analytic in that vicinity, and which can be made to satisfy the further conditions that y, z, y', z' shall assume arbitrarily prescribed values for $x = x_0$.

Let us consider four pairs of functions of x,

$$(y_1, z_1),\ (y_2, z_2),\ (y_3, z_3),\ (y_4, z_4),$$

which are simultaneous systems of solutions of (A). Then

$$(1)\qquad \begin{aligned} y &= c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4, \\ z &= c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4, \end{aligned}$$

where c_1, c_2, c_3, c_4 are arbitrary constants, will also form a simultaneous system of solutions. Moreover since

$$y' = c_1 y_1' + c_2 y_2' + c_3 y_3' + c_4 y_4',$$

$$z' = c_1 z_1' + c_2 z_2' + c_3 z_3' + c_4 z_4'$$

are equations simultaneous with equations (1), the constants c_1, c_2, c_3, c_4 can be determined in such a way as to give arbitrary constant values to y, z, y', z' for $x = x_0$ provided that

~~) Horn, Einführung in die Theorie der partiellen Differential-gleichungen, Art. 5, p. 18-25.

the determinant

$$D = \begin{vmatrix} y_1' & y_2' & y_3' & y_4' \\ z_1' & z_2' & z_3' & z_4' \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

is not equal to zero for $x = x_0$. If, therefore, $D \neq 0$, we can express a general system of solutions in terms of

$$y_1, \; y_2, \; y_3, \; y_4 \quad \text{and} \quad z_1, \; z_2, \; z_3, \; z_4$$

by means of (1); and the four pairs of solutions $(y_i, \; z_i)$ are a fundamental system of simultaneous solutions.

Under the transformations of the dependent variables

$$(2) \qquad y = \alpha\bar{y} + \beta\bar{z}, \quad z = \gamma\bar{y} + \delta\bar{z}, \quad \alpha\beta - \gamma y \neq 0,$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions of x the system (A) is transformed into another of the same form. Likewise a transformation

$$(3) \qquad x = f(\xi)$$

of the independent variable leaves the system unchanged in form ~).

~) See Wilczynski, Proj. Diff. Geom., p. 14.

Let us interpret y_1, y_2, y_3, y_4 and z_1, z_2, z_3, z_4 as the homogeneous coordinates of two points P_y and P_z of space. As x changes the point P_y describes the curve C_y and the point P_z describes the curve C_z. Moreover the points on the two curves C_y and C_z are put into one-to-one correspondence. Let us join two corresponding points by the straight line L_{yz}. As x assumes different values there is obtained a ruled surface S.

If in the transformations (2) we replace y by y_i and z by z_i, we obtain

$$(4) \qquad \bar{y}_i = \alpha y_i + \beta z_i \quad \bar{z}_i = \gamma y_i + \delta z_i \quad (i = 1,2,3,4)$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions of x. Such a transformation converts P_y and P_z of L_{yz} into two points $P_{\bar{y}}$ and $P_{\bar{z}}$ of the same line. Accordingly if L_{yz} is the generator of a ruled surface S, this transformation converts the curves C_y and C_z into any other two curves $C_{\bar{y}}$ and $C_{\bar{z}}$ upon this ruled surface. The correspondence of points still holds and C_y and C_z become new directrices of the ruled surface S. A transformation of the form $\xi = f(x)$, where $f(x)$ is an arbitrary function, changes the parametric representation without changing the curves or their point to point correspondence.

Thus, there belongs to every system of two linear homogeneous differential equations of the second order a ruled surface, whose generators are the lines joining the corresponding points of the two directrix curves. This ruled surface is the same for all such systems which can be transformed into each other by a transformation of the form

$$\bar{y} = \alpha y + \beta z, \quad \bar{z} = \gamma y + \delta z \quad \xi = f(x),$$

where $\alpha, \beta, \gamma, \delta, f$ are arbitrary functions of x.

However this ruled surface must not be developable.**) For the determinant D of equation (4) would then be zero. We have excluded this condition.

The surface S has been defined starting from a particular simultaneous fundamental system of solutions (y_i, z_i). But any four pairs of solutions $(\bar{y}_i, \bar{z}_i)$, obtained from the equations

$$(5) \quad \begin{aligned} \bar{y}_k &= c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4, \\ \bar{z}_k &= c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4, \end{aligned}$$

could have been taken as a fundamental system. A new surface S_1 would have been obtained. However equations (5) show that S and S_1 are projective transformations of each other. In general, if two systems of differential equations of form (A) can be transformed into each other by transformations of the form (5) their integrating ruled

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See Wilczynski, Proj. Diff. Geom., p. 130.

surfaces are projective transformations of each other.~)

Any fundamental system of solutions of (A) is valid only in the neighborhood of the point $x = x_0$, where the coefficients are regular. Therefore we can study the surface only in the vicinity of the generator determined by putting $x = x_0$. In other words our geometry is a projective, differential geometry.

Certain functions of the coefficients and their derivatives and the dependent variables and their derivatives are unchanged by the transformations (2)(3). Such functions have a significance for the ruled surface, not disturbed by any projective transformation and independent of the special method of representation. We say that these invariant combinations characterize the projective properties of the ruled surface.

Those functions of the coefficients and their derivatives which remain unchanged in form or value under the transformations (2) of the dependent variables are called seminvariants. If the functions involve the dependent variables or their derivatives they are called semi-covariants. If a seminvariant or a semi-covariant is unchanged in form or value by an arbitrary transformation of the independent variable it is termed an absolute invariant or an absolute covariant.

The seminvariants, invariants, semi-covariants, and covariants have been calculated for a general system of form (A) in Wil-

~) See Wilczynski, Proj. Diff. Geom., p. 132.

czynski's "Projective Differential Geometry of Curves and Ruled Surfaces," pp. 95 to 125.

It is there shown that all seminvariants are functions of certain fundamental seminvariants and their derivatives. Likewise all invariants are functions of certain fundamental invariants or of invariants obtained from them by the Jacobian process.~~)

II. Cubic Ruled Surfaces.

Non-developable ruled surfaces of the third order, or cubic scrolls, are of two types only, where it is understood that all surfaces which are projectively equivalent are of the same type. Cayley ~~) shows this and distinguishes the two types as $S(1,1,3)$ and $S(\overline{1,1},3)$, where $S(m,n,p)$ is defined as a ruled surface generated by a line which meets three directrix curves of the orders m, n, p respectively. Both types of the cubic surfaces have two straight line directrices each of which meets the generator in one point. Hence, for both types $m = 1$, $n = 1$. In the second type the straight line directrices coincide, a fact which Cayley indicates by the symbol $\overline{1,1}$. For the m-thic ruled surface there is a nodal curve of order $m - 2$, at least and of order $(m - 1)(m - 2)$ at most. Therefore for a cubic surface there is a

~~) See Wilczynski, Proj. Diff. Geom., pp. 121, 125.

~~) Cayley, A Second Memoir on Skew Surfaces, Otherwise Scrolls: Collected Mathematical Papers. Cambridge. 1892. Vol. v. pp. 201-213. Art. nos. 1 to 35. Philosophical Transactions of the Royal Society of London, Vol. cliv, pp. 559-576.

double straight line. Every cubic surface having a nodal or double line is a cubic ruled surface. For, any plane intersecting the surface in that line must intersect it in another line. Consequently there are on the surface a single infinity of lines, and the surface is a ruled surface.

For the canonical form of the equation of the two types of cubic ruled surfaces Cayley gives

$$(6)\qquad \begin{aligned} x^2z + y^2w &= 0, \\ x(yw - xz) + y^3 &= 0. \end{aligned}$$

To correspond with our notation we write the equations in the form

$$(6_a)\qquad x_1^2x_3 - x_2^2x_4 = 0$$

$$(6_b)\qquad x_1(x_2x_4 + x_1x_3) + x_2^3 = 0$$

The first cubic ruled surface has two straight line directrices whose parametric equations can be taken in the form

$$y_1 = y_2 = 0, \quad y_3 = x^2, \quad y_4 = 1;$$

$$z_1 = 1, \quad z_2 = x, \quad z_3 = z_4 = 0.$$

We must show that the line L_{yz} joining corresponding points on the two directrices lies on the surface. The homogeneous coordinates of any point on L_{yz} are given by

$$x_i = a y_i + b z_i \qquad (i = 1,2,3,4)$$

where a and b are arbitrary constants. When these values for x_i are substituted in (6_α) the equation is identically satisfied. Therefore L_{yz} can be taken as a generator.

The second cubic ruled surface has the directrices

$$y_1 = y_2 = 0, \quad y_3 = -x, \quad y_4 = -1;$$

$$z = -1, \quad z = -x, \quad z = 0, \quad z = x^2.$$

We show that the lines joining corresponding points on the two directrices lie on the surface just as in the previous case.

III. The Cubic Scroll S(1,1,3).

The first surface has two distinct straight line directrices. We have seen that their parametric equations can be written in the form

$$y_1 = y_2 = 0, \quad y_3 = x^2, \quad y_4 = 1;$$

$$z_1 = 1, \quad z_2 = x, \quad z_3 = z_4 = 0.$$

In order to form the differential equation for this ruled surface we set each pair of values for y and z in the general equation (A), and solve the resulting equations for p_{ik} and q_{ik}, $(i, k = 1,2)$. The substitutions in (A) give the equations

$$q_{12} = 0, \qquad q_{22} = 0,$$

$$p_{12} + x\, q_{12} = 0, \qquad p_{22} + x \cdot q_{22} = 0,$$

$$2 + 2x\, p_{11} + x^2 q_{11} = 0, \qquad 2x\, p_{21} + x^2 q_{21} = 0,$$

$$q_{11} = 0, \qquad q_{21} = 0,$$

from which we have

$$p_{11} = -\frac{1}{x}, \quad p_{12} = q_{11} = q_{12} = 0, \quad p_{21} = p_{22} = q_{21} = q_{22} = 0.$$

Therefore the differential equations of form (A) for this ruled surface are

$$\text{(I)} \qquad \begin{aligned} y'' - \frac{1}{x} y' &= 0, \\ z'' &= 0. \end{aligned}$$

In calculating the invariants Wilczynski makes use of several functions of the coefficients which he denotes by u_{ik}, v_{ik}, w_{ik}. They are given by the formulae

$$u_{ik} = 2p'_{ik} - 4q_{ik} + \sum_{j=1}^{2} p_{ij} p_{jk}, \qquad (i, k = 1, 2),$$

$$v_{ik} = 2u'_{ik} + \sum_{j=1}^{2} p_{ij} u_{jk} - p_{ik} u_{kj},$$

$$w_{ik} = 2v'_{ik} + \sum_{j=1}^{2} p_{ij} v_{jk} - p_{jk} v_{kj}.$$

For our system (I) they take the values

$$U_{11} = \frac{3}{X^2}, \quad U_{12} = U_{21} = U_{22} = 0,$$
$$V_{11} = \frac{-12}{X^3}, \quad V_{12} = V_{21} = V_{22} = 0,$$
$$W_{11} = \frac{72}{X^4}, \quad W_{12} = W_{21} = W_{22} = 0.$$

The fundamental seminvariants are

$$I = U_{11} + U_{22} = \frac{3}{X^2},$$
$$J = U_{11} U_{22} - U_{12} U_{21} = 0,$$
$$K = V_{11} V_{22} - V_{12} V_{21} = 0,$$
$$L = W_{11} W_{22} - W_{12} W_{21} = 0,$$

$$\Delta = \begin{vmatrix} \frac{3}{X^2}, & 0, & 0 \\ \frac{-12}{X^3}, & 0, & 0 \\ \frac{72}{X^4}, & 0, & 0 \end{vmatrix} = 0.$$

The fundamental invariants are

$$\Theta_4 = I^2 - 4J = \frac{9}{X^4},$$
$$\Theta_{4.1} = \left[\Theta_{m.1}\right]^{m=4} = \left[2m\Theta_m'' \Theta_m - (2m+1)(\Theta_m')^2 + \tfrac{1}{2} m^2 I \Theta_m^2\right]^{m=4}$$
$$= \frac{-648}{X^{10}},$$
$$\Theta_{10} = (I^2 - 4J)(K - I'^2) + (II' - 2J')^2 = 0,$$
$$\Theta_9 = \Delta = 0.$$

The semi-covariants from which all others may be obtained are

$$C = u_{12} z^2 - u_{21} y^2 + (u_{11} - u_{22}) y z = \frac{3}{x^2} y z ,$$

$$E = v_{12} z^2 - v_{21} y^2 + (v_{11} - v_{22}) y z = \frac{-12}{x^3} y z ,$$

$$P = 2(y'z - yz') + p_{12} z^2 - p_{21} y^2 + (p_{11} - p_{22}) y z = 2(y'z - yz') - \frac{1}{x} yz ,$$

$$G = 2C' + (p_{11} + p_{22}) C = -\frac{9}{x^3} yz + \frac{3}{x^2} (y'z + yz').$$

$$N = G - E = \frac{-3}{x^3} yz + \frac{6}{x^2} (y'z + yz').$$

All covariants may be expressed in terms of the covariants

$$C_1 = P = 2(y'z - yz') - \frac{1}{x} yz ,$$

$$C_2 = C = \frac{3}{x^2} yz ,$$

$$C_3 = E + 2N = -\frac{18}{x^3} yz + \frac{12}{x^2} (y'z + yz').$$

The semi-covariant P can be expressed in the form

$$P = z\rho - y\sigma$$

when ρ and σ are defined by

$$(7)\qquad \begin{aligned} \rho &= 2y' + p_{11}y + p_{12}z, \\ \sigma &= 2z' + p_{21}y + p_{22}z. \end{aligned}$$

The variables ρ and σ are cogredient with y and z ; i.e. if y and z are transformed by the equations

$$(2)\quad y = \alpha\bar{y} + \beta\bar{z}\,, \quad z = \gamma\bar{y} + \delta\bar{z}\,, \qquad \alpha\delta - \beta\gamma \neq 0,$$

then ρ and σ will be transformed by the equations

$$\rho = \alpha\bar{\rho} + \beta\bar{\sigma}\,, \quad \sigma = \gamma\bar{\rho} + \delta\bar{\sigma}\,.$$

By means of a transformation of the form

$$(2)\quad y = \alpha\bar{y} + \beta\bar{z}\,, \quad z = \gamma\bar{y} + \delta\bar{z}\,, \qquad \textit{where } \alpha\delta - \beta\gamma \neq 0,$$

every system of linear homogeneous differential equations of the form (A) may be converted into another system which involves no first derivatives of the dependent variable. The system is then said to be in the semi-canonical form. The proper values for α, β, γ, and δ are found by performing the substitutions and then solving the differential equations which arise from placing equal to zero the coefficient of the first derivative. Thus, the equations (I) become

$$\alpha y''+2\alpha' y'+\alpha'' y+\beta z''+2\beta' z'+\beta'' z-\frac{1}{x}\left[\alpha y'+\alpha' y+\beta z'+\beta' z\right]=0,$$
(8)
$$\gamma y''+2\gamma' y'+\gamma'' y+\delta z''+2\delta' z'+\delta'' z=0.$$

When we set the coefficients of y' and z' equal to zero we obtain the differential equations

$$2\alpha'-\frac{\alpha}{x}=0,\quad 2\beta'-\frac{\beta}{x}=0,\quad 2\gamma'=0,\quad 2\delta'=0.$$

On solving the equations we find

$$\alpha=\varepsilon\sqrt{x},\quad \beta=\varepsilon'\sqrt{x},\quad \gamma=c_1,\ \delta=c_2,$$ where $\varepsilon=\pm 1$ and $\varepsilon'=\pm 1$.

In order that $\alpha\delta-\beta\gamma\neq 0$ we must have $c_1\neq c_2$.

For example, let $c_1=1$, $c_2=4$. The substitution of the values for α, β, γ, δ in (8) gives the equations

$$\sqrt{x}\,y''-\frac{3}{4x^{\frac{3}{2}}}y-\frac{3}{4x^{2}}z+\sqrt{x}\,z''=0,$$

$$y''+4z''=0.$$

These may be combined into the semi-canonical form

$$y''-\frac{1}{x^2}y-\frac{1}{x^2}z=0,$$
(I')
$$z''+\frac{1}{4x^2}y+\frac{1}{4x^2}z=0.$$

When the system has been transformed the directrices are also transformed. The value of any transformed variable in terms of the original variable may be found by solving

$$y = \varepsilon \sqrt{X}\,\bar{y} + \varepsilon' \sqrt{X}\,\bar{z}$$

$$z = \bar{y} + 4\bar{z}$$

for y and z. We find

$$y_i = \frac{4 y_i - \varepsilon' \sqrt{X}\, z_i}{3 \varepsilon \sqrt{X}},$$

$$z_i = \frac{-y + \varepsilon \sqrt{X}\, z}{3 \varepsilon \sqrt{X}},$$

where $(i = (1, 2, 3, 4))$.

The parametric equations of the new directrices are given by

$$y_1 = -\frac{1}{3}, \qquad z_1 = \frac{1}{3},$$

$$y_2 = -\frac{X}{3}, \qquad z_2 = \frac{X}{3},$$

$$y_3 = \frac{4}{3}\varepsilon X^{\frac{3}{2}}, \qquad z_3 = -\frac{\varepsilon}{3} X^{\frac{3}{2}},$$

$$y_4 = \frac{4}{3\varepsilon\sqrt{X}}; \qquad z_4 = -\frac{1}{3\varepsilon\sqrt{X}}.$$

In order to find the geometrical significance of the semi-canonical form let us consider equations (A) with $p_{ik}=0$,

$$(\mathrm{e}) \qquad \begin{aligned} y_i'' + q_{11} y_i + q_{12} z_i &= 0, \\ z_i'' + q_{21} y_i + q_{22} z_i &= 0. \end{aligned} \qquad (i=1,2,3,4)$$

When $(y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)$ are interpreted as the homogeneous coordinates of two points, the point $P_{y''}$ determined by the coordinates $(y_1'', y_2'', y_3'', y_4'')$, , lies on the osculating plane of the curve C_y at P_y . The point P_η determined by the coordinates $\eta_i = q_{11} y_i + q_{12} z_i$, $(i=1,2,3,4)$ is a point on the generator through P_y and P_z . But equations (e) show that the points $P_{y''}$ and P_η coincide. Therefore when the equations are in the semi-canonical form the directrix curves are such that their osculating planes and the tangent plane to the surface at the same point coincide. In other words the directrix curves are asymptotic lines.

Wilczynski $^{\sim\sim}$) shows that the most general transformation of the dependent variable which leaves the semi-canonical form in the semi-canonical form is given by the equations

$$\bar{y} = ay + bz, \quad \bar{z} = cy + dz, \qquad ad - bc \neq 0,$$

$^{\sim\sim}$)See Wilczynski, Proj. Diff. Geom., p.115

where a, b, c, d are arbitrary constants. The most general transformation of the independent variable, $\xi = f(x)$, leaves it in the semi-canonical form.

When the equations (A) are in the semi-canonical form we see that each of the new directrices is an asymptotic curve of the surface. Any asymptotic curve of the surface is given by

$$x_i = a\,\bar{y}_i + b\,\bar{z}_i$$

when a and b are constants. We shall show that any asymptotic curve is quartic by finding the number of its intersections with the plane

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0.$$

For x_i substitute $a\,\bar{y}_i + b\,\bar{z}_i$ where $(i = 1, 2, 3, 4)$ and we have

$$a_1\left(-\frac{a}{3} + \frac{b}{3}\right) + a_2\left(-\frac{x}{3}a + \frac{x}{3}b\right) + a_3\left(\frac{4}{3}\varepsilon x^{\frac{3}{2}} a - \frac{\varepsilon}{3} x^{\frac{3}{2}} b\right) + a_4\left(\frac{4}{3\varepsilon\sqrt{x}} - \frac{1}{3\varepsilon\sqrt{x}}\right) = 0.$$

or

$$a_1(a-b)\varepsilon\sqrt{x} + a_2(a-b)\varepsilon\, x^{\frac{3}{2}} - a_3(4a-b)x^2 - a_4(4a-b) = 0$$

or

$$a_1^2(a-b)^2 X + 2a_1a_2(a-b)^2X^2 + a_2^2(a-b)^2X^3 - a_3^2(4a-b)^2X^4 - 2a_3a_4(4a-b)^2X - a_4^2(4a-b)^2 = 0$$

We see that this equation is of the fourth degree in X, so that there are four intersections with the plane.

It is possible to consider the ruled surface as expressed in line - coordinates instead of considering it in point - coordinates. We shall use the Plückerian line - coordinates, ω_{ik}, $(i, k = 1, 2, 3, 4)$

$$\omega_{ik} = y_i z_k - y_k z_i$$

where y_i and z_i are the coordinates of two points on a generator. Since $\omega_{ii} = 0$, and $\omega_{ik} = -\omega_{ik}$, we need retain only six of these quantities, say

$$\omega_{12},\ \omega_{13},\ \omega_{14},\ \omega_{23},\ \omega_{42},\ \omega_{34}.$$

We define these to be the six homogeneous coordinates of the line. There is a one-to-one correspondence between the lines of space and the ratios of the above six quantities. There is one relation between these six quantities. It may be taken in the form

$$\omega_{12}\omega_{34} + \omega_{13}\omega_{42} + \omega_{14}\omega_{23} = 0.$$

In general the six line coordinates will satisfy a linear homogeneous differential equation of the sixth order. When there are two linear relations between the coordinates, the differential equations reduce to the fourth order. We see that our surface has two such relations for it has two straight line directrices and therefore belongs to a linear congruence. In general ∼∼), a surface belongs to a linear congruence with distinct directrices if all the minors of the second order in Δ vanish while $\Theta \neq 0$. We have seen that these conditions are fulfilled in the case of our surface, and we know that its equation in the line-coordinates is at most of the fourth order.

Let (y, z) and (η, ζ) be any two simultaneous systems of solutions of the system (I_1') so that

$$
(10)\qquad
\begin{aligned}
y'' &= -q_{11}y - q_{12}z = \frac{1}{x^2}y + \frac{1}{x^2}z,\\
z'' &= -q_{21}y - q_{22}z = -\frac{1}{4x^2}y - \frac{1}{4x^2}z,\\
\eta'' &= -q_{11}\eta - q_{12}\zeta = \frac{1}{x^2}\eta + \frac{1}{x^2}\zeta,\\
\zeta &= q_{21}\eta - q_{22}\zeta = -\frac{1}{4x^2}\eta - \frac{1}{4x^2}\zeta.
\end{aligned}
$$

Then put $\omega = y\zeta - z\eta = (y\zeta)$. We denote by the symbol $(a\ \beta)$ the expression $a\beta - \alpha b$, obtained from the term actually written by subtracting a corresponding term, in which the Greek and the

∼∼) See Wilczynski, Proj. Diff. Geom., p. 167.

Roman letters are interchanged. We wish to find the differential equation satisfied by ω. From equations (10) we find

$$(\eta y'') = \frac{1}{x^2}\eta(y+z) - y(\eta+\zeta) = -\frac{1}{x^2}\omega,$$

$$(\zeta y'') = \frac{1}{x^2}\zeta(y+z) - z(\eta+\zeta) = \frac{1}{x^2}\omega,$$

$$(\eta z'') = -\frac{1}{4x^2}\eta(y+z) + \frac{1}{4x^2}y(\eta+\zeta) = \frac{1}{4x^2}\omega,$$

$$(\zeta z'') = -\frac{1}{4x^2}\zeta(y+z) + \frac{1}{4x^2}z(\eta+\zeta) = -\frac{1}{4x^2}\omega.$$

Let us put

$$2v = \omega'' + (q_{11}+q_{22})\omega = \omega'' - \frac{3}{4x^2}\omega,$$

$$w = v'' - 2(q_{11}q_{22} - q_{12}q_{21})\omega + (q_{11}+q_{22})v = \frac{\omega^{(iv)}}{2} - \frac{3}{4x^2}\omega'' + \frac{3}{2x^3}\omega'.$$

Then by means of successive differentiation we find three equations involving $(\eta z'), (\zeta z'), (\eta y'), (\zeta y')$. They are

$$\omega' = -(\eta z') + (\zeta y'),$$

$$v' = -\frac{1}{x^2}(\eta z') - \frac{1}{x^2}(\zeta z') - \frac{1}{4x^2}(\eta y') - \frac{1}{4x^2}(\zeta y'),$$

$$w = \frac{2}{x^3}(\eta z') + \frac{2}{x^3}(\zeta z') + \frac{1}{2x^3}(\eta y') + \frac{1}{2x^3}(\zeta y').$$

To find the desired equation we must eliminate $(\eta z')$, $(\xi z')$, $(\eta y')$, $(\xi y')$. We have $w + \frac{2}{x} v' = 0$, or

$$w^{(iv)} + \frac{2}{x} w''' - \frac{3}{2x^2} w'' + \frac{21}{2x^3} w' + \frac{3}{x^4} w = 0,$$

for the equation of the surface in line-coordinates.

Let us again consider the quantities ρ and σ, which for the semi-canonical form become

$$\rho = 2y', \quad \sigma = 2z'. \qquad \text{see (7).}$$

If in these equations we replace y by y_i, and z by z_i, where $(i = 1, 2, 3, 4)$ we have

$$\rho_i = 2y_i', \quad \sigma_i = 2z_i.$$

Then ρ_i and σ_i may be taken as the homogeneous coordinates of a point upon the tangent to the asymptotic curves C_y and C_z respectively. Thus as a point moves along the line L_{yz} there is a point corresponding to it moving along the line $L_{\rho\sigma}$. This second point is on the line tangent at the first point to the asymptotic curve through that point. This tangent line describes a hyperboloid H which osculates S, i.e. it has three point contact with the surface S. There exists for each generator on S a single hyperboloid. The totality of the generators $L_{\rho\sigma}$, a double infinity, makes up the flecnode congruence of S, denoted by Γ.

The condition, $J = 0$, which obtains here, is the condition that S′, the derivative surface, shall be developable. So long as the independent variable is unchanged the derivative surface is unchanged. S′ remains a developable only so long as $J = 0$. The transformation ~) of the independent variable for which $J = 0$, must satisfy the differential equation

$$(11) \qquad 4\mu^2 + 2I\mu + J = 0.$$

But $J = 0$, and $\mu = \eta' - \frac{1}{2}\eta^2$. Therefore differential equation (11) becomes

$$\eta' - \tfrac{1}{2}\eta^2 = 0, \qquad \eta' - \tfrac{1}{2}\eta^2 = -\frac{3}{2x^2}, \qquad \text{where } \eta = \frac{\xi''}{\xi'},$$

But the derivative surface is determined by the values of η since ρ and σ are transformed into $\bar{\rho}$ and $\bar{\sigma}$, where

$$\rho = \frac{1}{\xi'}(\rho + \eta y), \qquad \sigma = \frac{1}{\xi'}(\sigma + \eta z).$$

Therefore there are two families of ∞' developable surfaces.

When the equations of our surface are in form (I) the curves

~) See Wilczynski, Proj. Diff. Geom., p. 176.

C_ρ and C_σ are given by

$$\rho = 2y' - \frac{1}{X}y,$$

$$\sigma = 2z'.$$

In this case the coordinates of $P_\rho(0,0,3X,\frac{1}{X})$ show that P_ρ moves along a curve C_ρ as X changes; but the coordinates of $P_\sigma(0,2,0,0)$ show that P_σ is a point. Thus the derivative surface, since it is a ruled surface with C_ρ and C_σ as directrices, is a cone with its apex at P_σ.

The flecnode curves are the loci of the points at which four point tangents may be drawn. If the curves C_y and C_z themselves are the two branches of the flecnode curve, system (A) is characterized by the conditions $u_{12} = u_{21} = 0$. We notice that for our system I the flecnode curves are the two straight line directrices. We wish to see what the flecnode curves become under a transformation of the form (2). The $\bar{u}_{12}$ and the $\bar{u}_{21}$ of the transformed system[~~]) are given by

$$\Delta\bar{u}_{12} = \beta\delta u_{11} + \delta^2 u_{12} - \beta^2 u_{21} - \beta\delta u_{22}$$

$$\Delta = \alpha\beta - \gamma\delta,$$

$$\Delta\bar{u}_{21} = -\alpha\gamma u_{11} - \gamma^2 u_{12} + \alpha^2 u_{21} + \alpha\gamma u_{22}.$$

When the transformed values are set equal to zero we have

$$\Delta\bar{u}_{12} = \beta\delta\frac{3}{X^2} = 0, \qquad \Delta\bar{u}_{21} = -\alpha\gamma\frac{3}{X^2} = 0.$$

[~~]) See Wilczynski, Proj. Diff. Geom., p. 103.

But $\alpha\delta - \beta\gamma$ must not be equal to zero, therefore either $\alpha = \delta = 0$, or $\beta = \gamma = 0$. Hence the curves C_y and C_z are transformed into $C_{\bar{y}}$ and $C_{\bar{z}}$ by means of

$$y = \alpha \bar{y} \qquad\qquad y = \beta \bar{z}$$

or

$$z = \delta \bar{z} \qquad\qquad z = \gamma y$$

But by this transformation C_y and C_z are transformed into themselves and the flecnode curves of our ruled surface are always coincident with the straight line directrices.

The principal surface of a congruence Γ is that surface for which Θ_4 is unity. Under the transformation of the independent variable $X = \xi(X)$ the value of the transformed Θ is given by

$$(12) \qquad \bar{\Theta}_4 = \frac{1}{(\xi')^4}\,\Theta_4 = \frac{1}{(\xi')^4}\,\frac{9}{X^4}\,.$$

When $\bar{\Theta} = 1$ the equation (12) becomes

$$\frac{1}{(\xi')^4}\,\frac{9}{X^4} = 1\,,$$

$$\xi' = \sqrt[4]{\frac{9}{X^4}} = \pm\frac{\sqrt{3}}{X}\,,$$

$$\xi = \pm\sqrt{3}\,\log X + C.$$

But the derivative surface S is changed only by η. and $\eta = \frac{\xi''}{\xi'} = -\frac{1}{X}$. Therefore the principal surface of Γ is determined by

$$\bar{\rho} = \frac{1}{g'}\left(\rho - \frac{1}{X} y\right), \quad \bar{\sigma} = \frac{1}{g'}\left(\sigma - \frac{1}{X} z\right).$$

Let us consider the covariant

$$C_3 = \frac{-18}{X^3} yz + \frac{12}{X^2}(y'z + yz').$$

For a general ruled surface C_3 is of the form

$$C_3 = \alpha z - \beta y$$

where

$$(13) \quad \begin{aligned} \alpha &= 2(u_{11} - u_{22})\rho + 4u_{12}\sigma + \tfrac{1}{2}(v_{11} - v_{22})y + v_{12}z, \\ \beta &= 4u_{21}\rho - 2(u_{11} - u_{22})\sigma + v_{21}y - \tfrac{1}{2}(v_{11} - v_{22})z. \end{aligned}$$

The covariant C_3, therefore, determines a ruled surface Σ, whose generator $L_{\alpha\beta}$ is obtained by joining the points P_α and P_β determined by (13). The generator is determined by the following construction~~).

Let P_y and P_z be the two flecnodes, supposed distinct, on a given generator of the ruled surface S, and let P_α and P_β be the points corresponding to P_y and P_z respectively upon the principal surface of the flecnode congruence of S. At P_y, as well as at P_z, three important lines intersect, viz.: the generator, the flecnode tangent, and the tangent to the flecnode curve. All of these are in the plane tangent to the surface S at their point of intersection. In each of these plane pencils we construct a fourth line, the harmonic conjugate of the generator

~~) See Wilczynski, Proj. Diff. Geom., p. 219.

with respect to the other two. Each of these lines meets the line joining the point of the principal surface, which corresponds to the flecnode considered, to the other flecnode. The line which joins the two points of intersection, P_α and P_β, obtained in this way is the generator of Σ which corresponds to the given generator of S.

The surface Σ is not dependent upon the choice of the independent variable. Therefore, for our ruled surface, since $U_{12}=U_{21}=0$, we may choose the independent value so that $U_{11} - U_{22} = 1$. For our surface the points P_α and P_β are determined by

$$\alpha = 2\rho$$

$$\beta = -2\sigma$$

Hence we see that the α and β of the Σ surface are the ρ and σ of the principal surface and therefore the Σ surface and the principal surface for our congruence coincide.

4. The Cubic Scroll $S(\overline{1,1},3)$.

The equations for the second cubic ruled surface may be found just as for the first one. The substitutions give the equations

$$q_{12}=0\,,\ p_{12}+Xq_{12}=0\,,\ p_{11}+Xq_{11}=0,\ 2Xp_{12}+q_{11}+X^2q_{12}^2=0,$$

$$q_{22}=0,\ p_{22}+Xq_{22}=0,\ p_{21}+Xq_{21}=0,\ 2+2Xp_{22}+q_{21}+X^2q_{22}=0.$$

Solving the equations we find

$$p_{11} = p_{12} = q_{11} = q_{12} = 0\,,\ p_{21} = 2x\,,\ p_{22} = 0,\ q_{21} = -2\,,\ q_{22} = 0.$$

Therefore the equations for this ruled surface are

$$(\mathrm{II})\qquad \begin{aligned} &y'' = 0 \\ &z'' + 2x\,y' - 2y = 0. \end{aligned}$$

The functions U_{ik}, V_{ik}, W_{ik} and the seminvariants, the semi-covariants, the invariants, and the covariants are calculated just as in the case of the other ruled surface, with the following results.

$$U_{11} = U_{12} = U_{21} = 12\,,\ U_{22} = 0,\ V_{11} = V_{12} = V_{21} = V_{22} = 0,\ W_{11} = W_{12} = W_{21} = W_{22} = 0;$$

$$I = J = K = L = 0\ ;\ \Theta_4 = \Theta_{4.1} = \Theta_9 = \Theta_{10} = 0\,;$$

$C = -12y^2$, $\qquad C_1 = P = 2(y'z - yz' - xy^2)$,

$E = 0$, $\qquad C_2 = C = -12y^2$,

$G = -24y$, $\qquad C_3 = E + 2N = -48y$.

$P = 2(y'z - yz') - 2xy^2$,

$N = -24y$.

The curves C_ρ and C_σ are given by

$$\rho = 2y' \ , \quad \sigma = 2z' + 2xy.$$

In this case C_ρ becomes a point $P_\rho(0,0,-2,0)$. C_σ is the curve generated by the point $P_\sigma(0,-2,-2x^2,6x)$; and the derivative surface is a cone with its apex at P_ρ .

In order to find the semi-canonical form for this system we shall use the transformations

$$y = \alpha \bar{y} + \beta \bar{z} \ , \quad z = \gamma \bar{y} + \delta \bar{z}$$

where

$$\alpha = 1, \quad \beta = 1, \quad \gamma = -\frac{x^2}{2} + 1, \quad \delta = \frac{-x^2}{2}.$$

The semi-canonical form is

$$(\mathrm{II}') \qquad \begin{aligned} y'' + 3y + 3z &= 0 \\ z'' - 3y - 3z &= 0 \end{aligned}$$

The transformed variables are given in terms of the original variables by the equations

$$\bar{y} = \frac{x^2}{2} y + z,$$

$$\bar{z} = \left(1 - \frac{x^2}{2}\right) y - z.$$

The new directrix curves are

$$\bar{y}_1 = -1,\ \bar{y}_2 = -x,\ \bar{y}_3 = \frac{-x^3}{2},\ \bar{y}_4 = \frac{3}{2}x^2;$$

$$\bar{z}_1 = 1,\ \bar{z}_2 = x,\ \bar{z}_3 = \frac{x^3}{2} - x,\ \bar{z}_4 = 1 - \frac{x^2}{2}.$$

The asymptotic curves of this surface are cubics. For, when $a\,y_k + b z_k$ where a and b are constants, are substituted for x_k in the general equation of the plane

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0$$

we have

$$a_1(-a+b) + a_2(-xa + xb) + a_3\left\{\frac{-x^3}{2}a + \left(\frac{x^3}{2} - x\right)b\right\} + a_4\left\{\frac{3}{2}x^2 a + \left(1 - \frac{x^2}{2}\right)b\right\} = 0,$$

or

$$2a_1(a-b) - 2a_4 b + 2\{a_2(a-b) + a_3 b\}x + a_4(b-3a)x^2 + a_3(a-b)x^3 = 0.$$

Since this equation is of the third degree in x, the asymptotic curves are cubic curves.

To find the equation of the ruled surface in line-coordinates we write the equations

$$\omega' = -(\eta z') + (\zeta y),$$

$$v' = 3(\eta z') + 3(\zeta z') + 3(\zeta y'),$$

$$w = 0,$$

where

$$V' = \frac{1}{2}\omega'''$$

$$W = \frac{1}{2}\omega^{(iv)}.$$

Thus the desired equation is $W=0$ or $\omega^{(v)}=0$.

This ruled surface belongs to a linear congruence with coincident directrices, for the minors of Δ vanish and $\Theta_4 = 0$

Since $J = 0$, the derivative surface is developable, and will remain developable for those transformations of the independent variable which satisfy the equation

$$(11) \qquad 4\mu^2 + 2I\mu + J = 0$$

But $J = I = 0$ and $\mu = \eta' - \frac{1}{2}\eta^2$. Therefore differential equation becomes

$$(\eta'' - \frac{1}{2}\eta^2)^2 = 0.$$

We see that this congruence contains two families of ∞' developable surfaces which coincide.

The flecnode curve determined by $U_{12} = 0$ coincides with the straight line directrix. Now $U_{21} \neq 0$ and the transformation, see p. 23, which would make $\Delta\, \bar{U}_{21} = 0$ is impossible, for such a transformation would require $\alpha = \beta = 0$. Hence we see that for this surface the flecnode curve has but one branch.

When $\Theta_4 = 0$ the constructions for the Σ surface and the principal surface of the congruence break down and these surfaces are indeterminate for the cubic ruled surface of the second type.